# Harmonic extension through cylindrical and conical surfaces

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The Schwarz reflection principle provides a simple formula for extending a harmonic function $h$ on a domain $\omega \subset \mathbb{R}^N$ through a relatively open subset $E$ of $\partial\omega$ on which $h$ vanishes, provided $E$ lies in a hyperplane. A related reflection formula holds when $E$ lies in a sphere. Indeed, when $N = 2$, a reflection formula holds whenever $E$ is contained in an analytic arc. However, when $N \geq 3$ and $N$ is odd, Ebenfelt and Khavinson [1] (cf. Chapter 12 of [8]) have shown that a point-to-point reflection law holds only when the containing real analytic surface is either a hyperplane or a sphere. Thus other ideas are needed to investigate the extension properties of harmonic functions that vanish on other types of set $E$.

In this direction the following problem was posed by Dima Khavinson at various international conferences: if $h$ is harmonic on an infinite cylinder and vanishes on the boundary, does it extend harmonically to all of $\mathbb{R}^N$? In the planar case, where $h$ is harmonic on an infinite strip, a positive answer follows readily by repeated application of the Schwarz reflection principle. In higher dimensions the problem was eventually also shown to have an affirmative answer [2] by analysis of the Green function of the cylinder. Thus we have a solution to one non-trivial case of the following general problem, where the set $E$ is assumed to lie in a real analytic surface.

**Problem 1** For a domain $\omega$ in $\mathbb{R}^N$ and a subset $E$ of $\partial\omega$ identify a larger domain $\omega_E$ such that each harmonic function on $\omega$ which vanishes continuously on $E$ has a harmonic extension to $\omega_E$.

This problem can be challenging even when $E$ is contained in the zero set of some polynomial. For example, the cylindrical case corresponds to the polynomial $(x', x_N) \mapsto \|x'\|^2 - 1$, where $x' \in \mathbb{R}^{N-1}$. This talk will describe several results in this direction where $E$ is contained in a cylindrical or conical surface. The various domain reflection results that arise are striking, given that reflection formulae for the harmonic functions themselves are known not to exist.

Currently, our methods rely on detailed calculations involving Bessel and Legendre functions. Our hope is that the outcome of these specific studies will suggest a more general approach to Problem 1.

This is joint work with Hermann Render ([2] - [7]).

## References


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